He's Variational Iteration Method for Solving a Partial Differential Equation Arising in Modelling of the Water Waves

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The variational iteration method is applied to solve the Kawahara equation. This method produces the solutions in terms of convergent series and does not require linearization or small perturbation. Some examples are given. The comparison with the theoretical solution shows that the variational iteration method is an efficient method.

Key words: Kawahara Equation; Travelling Wave Solution; Variational Iteration Method; Closed Form Solution.
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1. Introduction

The Kawahara equation is an equation which models the plasma waves and the capillary gravity water waves [1 – 3]. Moreover, this equation describes water waves with surface tension [4]. In this paper we consider the standard form of the Kawahara equation given by [5]:

$$u_t + uu_x + u_{3x} - u_{5x} = 0, \tag{1}$$

where $u_{kx} = \frac{\partial^k u}{\partial x^k}$. We solve (1) with subject to the initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}. \tag{2}$$

In [5] an algebraic direct method is described to construct the exact travelling wave solutions for this problem. Authors of [6] used Adomian decomposition method (ADM) and the Sinc-Galerkin method for numerical approximation of the travelling wave solutions of this equation and they obtained the exact travelling wave solutions of the Kawahara equation for the initial conditions by using ADM. Also ADM is used in [7] for numerical approximation of the travelling wave solutions of the modified Kawahara equation. Author of [8] demonstrated the existence of compaction solutions and solitary patterns solutions for two specific forms of the Kawahara-type equation.

In the current investigation, we employ the variational iteration method (VIM) for solving the Kawahara equation. This method was proposed by the Chinese mathematician Ji-Huan He [9, 10] as a modification of a general Lagrange multiplier method [11]. The VIM plays an important role in recent researches for solving various kinds of problems (see for example [12 – 16] and the references therein). This method does not require specific transformations for nonlinear terms as required by some other existing techniques.

An elementary introduction of VIM is given in [17]. The main concepts in variational iteration method, such as general Lagrange multiplier, restricted variation, and correction functional are explained heuristically. Subsequently, the solution procedure is systematically addressed, in particular for nonlinear oscillators.

Using the VIM we can find the exact solution of the problem. As Wazwaz said in [18] in the variational iteration method (VIM), the linear and nonlinear structures are handled in a like manner without any need to restrictive assumptions. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. As an advantage of the VIM over the ADM, the former method provides the solution of the problem without calculating Adomians polynomials. This technique solves the problem without any need to discretization of the variables. Therefore, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. This scheme provides the solution of the problem in a closed form while the mesh point techniques [19] provide the approximation at mesh points only. Also this procedure is useful for finding an accurate approximation of the exact solution. The He's variational iteration method gives several successive approximations through using the iteration of the correction functional. The reliability of the method and the reduction of the size of computational domain have attracted the attention of many researchers to use it in various areas of science and engineering. With our best of knowledge [20] is the only paper which has investigated the convergence of the variational iteration method.

Recently, some other extraordinary virtues of the method have been exploited and wide applications have been found in various fields. The variational iteration method was employed in [21] to solve the Cauchy problem arising in one-dimensional nonlinear thermoelasticity. The advantages of the technique to overcome the difficulty of calculation of Adomian's polynomials in the decomposition procedure of Adomian is shown. He's variational iteration method is used in [22] to solve various integral equations. The results show that VIM is very effiective and convenient for solving integral equations. Wazwaz [23] employed VIM to present an analytic frame work to handle the linear and the nonlinear Goursat problems. Authors of [24] used He's variational iteration method to find the solution of nonlinear Jaulent-Miodek equation, Korteweg-de Vries (KdV), and modified Korteweg-de Vries (MKdV) equations. All the examples show that the results of the VIM are in excellent agreement with those of ADM . They showed that VIM with the fewest number of iterations or even in some cases, once, can converge to correct results. Abbasbandy [25] solved the quadratic Riccati differential equation by He's variational iteration method with considering Adomian's polynomials. Comparisons were made between Adomian's decomposition method, homotopy perturbation method, and the exact solution. Authors of [26] employed this procedure to solve the Klein-Gordon equation which is the relativistic version of the Schrödinger equation. The variational iteration technique is used in [27] to solve a system of two nonlinear integro-differential equations which arises in biology, describing biological species living together. VIM is employed by authors of [28] to solve a parabolic integro-differential equation which describes problems in heat conduction in materials with memory. This scheme is applied to solve the Lane-Emden equation which arises in astronomy [29]. Authors of [30] employed the variational iteration procedure to solve the Cauchy reaction-diffusion equation.

The main analytic approach in the literature is the Adomian decomposition method [31]. But the main disadvantage of the Adomian method is that the solution procedure for calculation of Adomian polynomials is complex and difficult as pointed by many researchers. A complete comparison between the Adomian decomposition method and the variational iteration method is available on [18]. Also some new developments and new interpretations of the variational iteration method is available on [17]. For a relatively comprehensive survey on the method and its applications, the readers are referred to the review article [32]. Also authors of [33] used He's variational iteration method to find the solution of a generalized pantograph equation. The interested reader can see [34 – 40] for more applications of the variation iteration method. The organization of this paper is as follows:

In Section 2, we apply the VIM on (1) with initial condition (2). To present a clear overview of the method, in Section 3 we give two examples with analytical solutions, and solve them using the variational iteration technique. A brief conclusion is given in Section 4.

2. The Application of VIM

In this section the application of the VIM is discussed for solving problem (1)-(2). As stated before, the variational iteration method is based on the general Lagranges multiplier method. To illustrate the basic concept of VIM, we consider the following general nonlinear differential equation:

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is a known analytical function. According to VIM we can construct a correction functional as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &+ \int_0^x \lambda(s)\{Lu_n(s) + N(\tilde{u}_n(s)) - g(s)\}\,\mathrm{d}s, \quad n \geq 0, \end{aligned} \tag{4}$$

where λ is a general Lagrangian multiplier [11] which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{u}_n$ is considered as a restricted variation, i. e.

$\delta\tilde{u}_n = 0$ [9, 10]. Therefore, we first determine the Lagrange multiplier λ which will be identified optimally via integration by parts. The successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrangian multiplier and any selective function u_0. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

Now, according to the VIM, we consider the correction functional in t direction for (1) in the following form:

$$\begin{aligned} u_{n+1}(x,t) = \\ u_n(x,t) + \int_0^t \lambda \Big\{ \frac{\partial u_n(x,s)}{\partial s} + u_n(x,s)\frac{\partial \tilde{u}_n(x,s)}{\partial x} \\ + \frac{\partial^3 \tilde{u}_n(x,s)}{\partial x^3} - \frac{\partial^5 \tilde{u}_n(x,s)}{\partial x^5} \Big\} \mathrm{d}s, \quad n \geq 0. \end{aligned} \tag{5}$$

To find the optimal value of λ we have

$$\begin{aligned} \delta u_{n+1}(x,t) = \delta u_n(x,t) \\ + \delta \int_0^t \lambda \Big\{ \frac{\partial u_n(x,s)}{\partial s} + u_n(x,s)\frac{\partial \tilde{u}_n(x,s)}{\partial x} \\ + \frac{\partial^3 \tilde{u}_n(x,s)}{\partial x^3} - \frac{\partial^5 \tilde{u}_n(x,s)}{\partial x^5} \Big\} \mathrm{d}s, \end{aligned} \tag{6}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda \left\{ \frac{\partial u_n(x,s)}{\partial s} \right\} \mathrm{d}s, \tag{7}$$

that results in

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1+\lambda) - \int_0^t \delta u_n(x,t)\lambda' \mathrm{d}s, \tag{8}$$

which yields

$$\lambda'(s) = 0, \quad 1 + \lambda(s) = 0|_{s=t}. \tag{9}$$

Thus we have $\lambda = -1$ and we can obtain the following iteration formula:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) \\ - \int_0^t \Big\{ \frac{\partial u_n(x,s)}{\partial s} + u_n(x,s)\frac{\partial u_n(x,s)}{\partial x} \\ + \frac{\partial^3 u_n(x,s)}{\partial x^3} - \frac{\partial^5 u_n(x,s)}{\partial x^5} \Big\} \mathrm{d}s, \quad n \geq 0. \end{aligned} \tag{10}$$

Therefore, the approximation solution is given as $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

3. Numerical Examples

To incorporate our discussion given in the previous section, we consider two examples with known exact solutions [5].

Table 1. Errors $|u_3(x_i,t_j) - u(x_i,t_j)|$ for Example 1.

x_i	$t_j = 0.1$	0.2	0.3	0.4	0.5
0.1	$0.10\cdot10^{-7}$	$0.60\cdot10^{-7}$	$0.21\cdot10^{-6}$	$0.67\cdot10^{-6}$	$0.16\cdot10^{-5}$
0.2	0	$0.20\cdot10^{-7}$	$0.18\cdot10^{-6}$	$0.60\cdot10^{-6}$	$0.15\cdot10^{-5}$
0.3	0	$0.20\cdot10^{-7}$	$0.18\cdot10^{-6}$	$0.58\cdot10^{-6}$	$0.14\cdot10^{-5}$
0.4	0	$0.40\cdot10^{-7}$	$0.17\cdot10^{-6}$	$0.56\cdot10^{-6}$	$0.13\cdot10^{-5}$
0.5	0	$0.40\cdot10^{-7}$	$0.18\cdot10^{-6}$	$0.52\cdot10^{-6}$	$0.12\cdot10^{-5}$

3.1. Example 1

We first consider the Kawahara equation which has the travelling wave solution of which is to be obtained subject to the initial condition [5, 6]

$$u(x,0) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4(Kx),$$

where $K = \frac{1}{2\sqrt{13}}$.

We start with an initial approximation $u_0(x,t) = u(x,0)$ and we use the iteration formula (10). We can obtain directly the other components as

$$u_0(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4(Kx). \tag{11}$$

$$\begin{aligned} u_1(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4(Kx) \\ -\frac{420}{28561}\operatorname{sech}(Kx)Kt\Big[72\cosh^4(Kx) - 105 \\ -2704K^2\cosh^4(Kx) + 5070K^2\cosh^2(Kx) \\ +43264K^2\cosh^4(Kx) - 263640K^4\cosh^2(Kx) \\ +283920K^4\Big]\frac{1}{\cosh^9(Kx)}, \end{aligned} \tag{12}$$

and so on, in the same manner the rest of components of the iteration formula (10) were obtained using the well-known symbolic package Maple. Using Taylor series, we obtain the closed form of the solution:

$$u(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4(K(x+ct)), \tag{13}$$

where $c = \frac{36}{169}$. The results are the same as with the results of the Adomian decomposition method [6]. The absolute errors obtained for $u_3(x,t)$ for different values of x_i and t_j are shown in Table 1.

3.2. Example 2

In this example [5] we consider the Kawahara equation (1) which has the travelling wave solution. Sup-

Table 2. Errors $|u_3(x_i,t_j)-u(x_i,t_j)|$ for Example 2.

x_i	$t_j=0.1$	0.2	0.3	0.4	0.5
0.1	$0.28\cdot10^{-8}$	$0.40\cdot10^{-7}$	$0.20\cdot10^{-6}$	$0.63\cdot10^{-6}$	$0.15\cdot10^{-5}$
0.2	$0.28\cdot10^{-8}$	$0.39\cdot10^{-7}$	$0.19\cdot10^{-6}$	$0.62\cdot10^{-6}$	$0.15\cdot10^{-5}$
0.3	$0.28\cdot10^{-8}$	$0.38\cdot10^{-7}$	$0.19\cdot10^{-6}$	$0.60\cdot10^{-6}$	$0.14\cdot10^{-5}$
0.4	$0.17\cdot10^{-8}$	$0.36\cdot10^{-7}$	$0.18\cdot10^{-6}$	$0.56\cdot10^{-6}$	$0.13\cdot10^{-5}$
0.5	$0.20\cdot10^{-8}$	$0.34\cdot10^{-7}$	$0.17\cdot10^{-6}$	$0.53\cdot10^{-6}$	$0.13\cdot10^{-5}$

pose

$$u(x,0)=-\frac{72}{169}+\frac{420}{169}\frac{\operatorname{sech}^2(Kx)}{(1+\operatorname{sech}(Kx))^2}, \tag{14}$$

where $K=\frac{1}{\sqrt{13}}$.

Again we start with an initial approximation $u_0(x,t)=u(x,0)$ and by substituting this equation into (10) we have

$$\begin{aligned}u_1(x,t)=&-\frac{72}{169}+\frac{420}{169}\frac{\operatorname{sech}^2(Kx)}{(1+\operatorname{sech}(Kx))^2}\\&-\frac{840}{28561}\sinh(Kx)Kt\Big[2704K^4\cosh(Kx)^2\\&-676K^2\cosh(Kx)^2+72\cosh(Kx)^2\\&-27547K^4\cosh(Kx)-1183K^2\cosh(Kx)\\&+144\cosh(Kx)-348+40729K^4\\&+1859K^2\Big]\frac{1}{1+\cosh(Kx)^5}.\end{aligned} \tag{15}$$

The remaining $u_n(x,t)$, $n\geq 2$, can be completely determined using the Maple package. Again using Taylor series, we obtain the closed form solution

$$u(x,t)=-\frac{72}{169}+\frac{420}{169}\frac{\operatorname{sech}^4(K(x+ct))}{(1+\operatorname{sech}(K(x+ct)))^2}. \tag{16}$$

In Table 2 the absolute errors obtained for $u_3(x,t)$ for different values of x_i and t_j are shown.

4. Conclusion

In this work, He's variational iteration method has been successfully applied to solve the Kawahara equation. The approximate solutions are compared with the exact solutions in Tables 1 and 2. This method solves the problem without any need to discretization of the variables. Therefore, it is not affected by computation round off errors. Also this method is useful for finding an accurate approximation of the exact solution. As an advantage of the variational iteration method over the decomposition procedure of Adomian, the present method provides the solution of the problem without calculating Adomians polynomials. In our work, we used the Maple package to calculate the terms of the series obtained from the variational iteration method. Overall, the reliability of the method and the reduction of the size of computational domain give this method wide applications.

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